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ORIGINAL PAPER

Numerical solution of ordinary differential equations by Fluctuationlessness theorem

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Abstract This paper presents a numerical method based on Fluctuationlessness Theorem for the solution of Ordinary Differential Equations over appropriately defined Hilbert Spaces. We focus on the linear differential equations in this work. The approximated solution is written in the form of an nth degree polynomial of the independent variable. The unknown coefficients are obtained by setting up a system of linear equations which satisfy the initial or boundary conditions and the differential equation at the grid points, which are constructed as the independent variable's matrix representation restricted to an n dimensional subspace of the Hilbert Space. An error comparison of the numerical solution and the MacLaurin series with the analytical solution is performed. The results show that the numerical solution obtained here converges to the analytical solution without using too many mesh points.

Keywords Ordinary differential equations · Fluctuations · Hilbert spaces · Eigenvalues

1 Introduction

Differential Equations have a great importance for the modeling of many problems from various scientific and engineering problems in Applied Mathematics [1]. These problems may contain either partial or ordinary differential equations. Here we focus on ordinary differential equations. Although it may be considered too simple, we aim

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to develop a method to have a heuristical point of view to construct a strong basis for the future theory. There is a huge accumulation knowledge for solving the ordinary differential equations both analytically and numerically [2,3]. Since not every problem can be solved analytically, scientists have developed iterative numerical solutions [4–8]. These methods may have good approximations to the analytical solutions but they may need to use too many mesh points, which may bring computational complexity for the solution [9]. But the method that we develop here have a rapid numerical convergence even by using a few mesh points.

Fluctuations may arise in probabilistic events and they describe random plus or minus deviations from means. In these cases, the individual components' behaviors can not be determined in the framework of causality. Instead, the evolution of the probability can be determined. In Quantum Mechanics [10–15], Non-equilibrium Statistical Mechanics [16] and Quantum Chemistry [17–19], fluctuations have a lot of importance. We don't get into the details of the fluctuations and their roles in these kinds of phenomena here, instead we will deal with the case of fluctuationlessness in mathematical sense. We will work in an appropriately chosen Hilbert Space throughout the paper.

This paper is organized as follows: In the second section we explain Fluctuationlessness Theorem in detail. In the third section the application of this theorem in numerical solution for an arbitrary order of linear Ordinary Differential Equations is explained. The fourth section covers the implementations presented in tables comparatively. The fifth section completes the paper via concluding remarks.

2 Fluctuationlessness theorem

Let $\mathcal{H}$ be a Hilbert Space generated by analytic and square integrable univariate functions on a closed interval $[a, b]$ and let $\mathcal{H}_n$ be a subspace generated by orthonormal functions $u_1(x), \ldots, u_n(x)$. We define the inner product of functions $f(x)$ and $g(x)$ in this space as

$$(f, g) \equiv \int_a^b f(x)g(x)w(x)dx, \tag{1}$$

where $w(x)$ is a given weight function. We can express a function $g(x)$ in this space as a linear combination of above basis functions as follows:

$$g\,(x) \equiv \sum_{i=1}^{n} g_i u_i\,(x) \tag{2}$$

Here coefficients g_i, $(1 \leq i \leq n)$ are real constants and depend on the structure of the function $g(x)$. This dependency can be determined by orthonormality of the basis functions. Therefore we can write the inner product of u_k and g in the following way:

$$(u_k, g) \equiv \left(u_k, \sum_{i=1}^{n} g_i u_i \right) = \sum_{i=1}^{n} g_i \left(u_k, u_i \right) = \sum_{i=1}^{n} g_i \delta_{k,i} = g_k, \quad 1 \le k \le n \quad (3)$$

If we substitute this result in (2) we obtain the following equation for the function $g(x)$:

$$g(x) = \sum_{i=1}^{n} u_i(x)(u_i, g) \equiv \sum_{i=1}^{n} P_i g(x) \equiv P^{(n)} g(x) \quad (4)$$

Here P_i is an integral operator which projects to the space which is spanned by $u_i(x)$. The operator represented by $P^{(n)}$ projects to an n-dimensional space spanned by $u_1(x), \ldots, u_n(x)$. While $P^{(n)}$ is a unitary operator on this space, it is a projection operator in $\mathcal{H} - \mathcal{H}_n$.

Now we define a new operator $\widehat{x}$. The domain of this operator is $\mathcal{H}$ and the action of this operator on a function $g(x)$ in $\mathcal{H}_n$ can be expressed as follows:

$$\widehat{x} g(x) = x g(x), \quad x \in [a, b] \quad (5)$$

This operator is an algebraic operator [20]. Its function is to multiply its operand by x. Now we will obtain the matrix representation of $\widehat{x}$ operator. We can write the following equation for $g(x)$ given in (2):

$$\widehat{x} g(x) = \sum_{j=1}^{n} g_j \widehat{x} u_j(x) = \widehat{x} P^{(n)} g(x) \quad (6)$$

We can observe that although $P^{(n)} g(x)$ is in the space spanned by $u_1(x), \ldots, u_n(x)$, the operator obtained multiplying by x may not be in this space. Therefore the effect of $\widehat{x}$ operator on a function in an n-dimensional space will cause a space extension. In order to avoid from this situation, we will use $P^{(n)}\widehat{x}$ operator instead of $\widehat{x}$, which is the projection of $\widehat{x}$ operator on this n-dimensional space. Hence we obtain:

$$P^{(n)} \widehat{x} g(x) = \sum_{j=1}^{n} g_j P^{(n)} \widehat{x} u_j(x) = P^{(n)} \widehat{x} P^{(n)} g(x) \quad (7)$$

Here we introduce an operator $\widehat{x}_{res}$ which is defined as follows:

$$\widehat{x}_{res} \equiv P^{(n)} \widehat{x} P^{(n)} \quad (8)$$

Now we can construct the matrix representation of $\widehat{x}_{res}$ operator by the following procedures. First we define a new function $h(x)$ as

$$h(x) \equiv P^{(n)} \widehat{x} P^{(n)} = \sum_{j=1}^{n} g_j P^{(n)} \widehat{x} P^{(n)} u_j(x). \quad (9)$$

Since this function is in the space spanned by $u_1(x), \ldots, u_n(x)$, we can write the following expressions:

$$h(x) = \sum_{k=1}^{n} h_k u_k(x) \tag{10}$$

and

$$\sum_{k=1}^{n} h_k u_k(x) = \sum_{j=1}^{n} g_j P^{(n)} \widehat{x} P^{(n)} u_j\,(x) \tag{11}$$

If we take the inner product of the both sides in (11) by $u_i\,(x)$, $(1 \le i \le n)$ and use the orthonormality property, we obtain the following equation:

$$h_i = \sum_{j=1}^{n} \left(u_i, P^{(n)} \widehat{x} P^{(n)} u_j \right) g_j, \quad 1 \le i \le n \tag{12}$$

We express this equation in terms of n-dimensional vectors $\mathbf{g}$ and $\mathbf{h}$, and the matrix $\mathbf{X}$ whose general term is $X_{i,j}^{(n)}$.

$$\mathbf{g} = [g_1 \ldots g_n]^T, \quad \mathbf{h} = [h_1 \ldots h_n]^T, \tag{13}$$

$$X_{i,j}^{(n)} = \left(u_i, P^{(n)} \widehat{x} P^{(n)} u_j \right), \quad 1 \le i, j \le n \tag{14}$$

Therefore we can express (12) in terms of (13) and (14) as

$$\mathbf{h} = \mathbf{X}^{(n)} \mathbf{g}. \tag{15}$$

To obtain this equation in a compact form, we explained the procedure of obtaining the matrix representation of $\widehat{x}_{res}$ operator in detail although these concepts are well known. Here the matrix $\mathbf{X}^{(n)}$ is the matrix representation of $\widehat{x}_{res}$ defined from $\mathcal{H}_n$ to $\mathcal{H}_n$. Now we can write the following equation for $\widehat{x}$ operator.

$$\begin{aligned} \widehat{x} &\equiv \left(P^{(n)} + \left[\widehat{I} - P^{(n)} \right] \right) \widehat{x} \left(P^{(n)} + \left[\widehat{I} - P^{(n)} \right] \right) \\ &= P^{(n)} \widehat{x} P^{(n)} + \left[\widehat{I} - P^{(n)} \right] \widehat{x} P^{(n)} \\ &\quad + P^{(n)} \widehat{x} \left[\widehat{I} - P^{(n)} \right] + \left[\widehat{I} - P^{(n)} \right] \widehat{x} \left[\widehat{I} - P^{(n)} \right] \end{aligned} \tag{16}$$

If we define $\widehat{x}_{fluc}$ as

$$\widehat{x}_{fluc} \equiv \left[\widehat{I} - P^{(n)} \right] \widehat{x} P^{(n)} + P^{(n)} \widehat{x} \left[\widehat{I} - P^{(n)} \right] + \left[\widehat{I} - P^{(n)} \right] \widehat{x} \left[\widehat{I} - P^{(n)} \right], \tag{17}$$

then we can write the following equation for $\widehat{x}$ operator:

$$\widehat{x} \equiv \widehat{x}_{res} + \widehat{x}_{fluc} \tag{18}$$

Here $\left[\widehat{I} - P^{(n)}\right]$ operator is a projection operator which goes to $\widehat{0}$ operator as n goes to infinity. The matrix representation of this operator in the space spanned by $u_1(x), \ldots, u_n(x)$ is the $\mathbf{0}$ matrix. But the infinite dimensional matrix representation of this operator in the Hilbert Space $\mathcal{H}$ is different than the infinite dimensional zero matrix although its norm gets smaller as n increases. This situation can be explained by the interaction of $\mathcal{H}_n$ and $\widehat{x}$ operator. The indices of the basis functions in $\left[\widehat{I} - P^{(n)}\right]$ operator are greater than n. The orthonormality property of the basis functions causes oscillations at the variables of the basis functions. In fact u_n has exactly n zeroes in the orthonormality domain. This means oscillations between positive and negative values. The frequency of these oscillations increase as n grows and this situation causes great cancelations at the output terms of the action of the operator $\left[\widehat{I} - P^{(n)}\right]$ on any function in $\mathcal{H}$. In other words, the image of a function in $\mathcal{H}$ under this operator fluctuates around zero and somehow calculates the oscillations around zero. This operator is called as nth order Fluctuation Operator [20,21].

The above procedures can also be applied to the square of $\widehat{x}$ operator, $\widehat{x}^2$. In this case the matrix representation of $\widehat{x}^2_{fluc}$ is not the zero matrix. In fact it is equal to the matrix representation of $P^{(n)}\widehat{x}\left[\widehat{I} - P^{(n)}\right]\widehat{x}P^{(n)}$ operator. This can be proven via these equations:

$$P^{(n)^2} = P^{(n)}, \quad P^{(n)}\left[\widehat{I} - P^{(n)}\right] = \widehat{0}_n, \tag{19}$$

$$\left[\widehat{I} - P^{(n)}\right]P^{(n)} = \widehat{0}_n, \quad \left[\widehat{I} - P^{(n)}\right]^2 = \left[\widehat{I} - P^{(n)}\right] \tag{20}$$

The matrix representation of $P^{(n)}\widehat{x}\left[\widehat{I} - P^{(n)}\right]\widehat{x}P^{(n)}$ operator on an n dimensional space is equal to the subtraction of the square of the matrix representation of the operator $P^{(n)}\widehat{x}P^{(n)}$ from the matrix representation of the operator $P^{(n)}\widehat{x}^2P^{(n)}$. This term defines the dominant contribution resulting from oscillations. Therefore the term $P^{(n)}\widehat{x}\left[\widehat{I} - P^{(n)}\right]\widehat{x}P^{(n)}$ is called as Independent Variable's First Order Fluctuation Operator, since the fluctuation operator $\left[\widehat{I} - P^{(n)}\right]$ appears once in this term [20]. The approximation obtained by ignoring the terms which contain fluctuation operator is called as Fluctuationlessness Approximation. In this case $\widehat{x}$ operator can be expressed in fluctuationlessness limit as

$$\widehat{x} \approx \widehat{x}_{res} \equiv P^{(n)}\widehat{x}P^{(n)}. \tag{21}$$

Now we will define a function type operator $\widehat{f}$, that multiplies its operand by an analytic function $f(x)$ defined on a closed interval $[a, b]$. The domain of this operator is the Hilbert Space of continuous and square integrable functions on $[a, b]$. This operator is an algebraic operator which multiplies its operand by $f(x)$. This can be expressed as

$$\widehat{f}g(x) = f(x)g(x), \quad x \in [a, b]. \tag{22}$$

Hence $\widehat{f}$ is an algebraic operator and it can be defined in terms of $\widehat{x}$ as follows:

$$\widehat{f} \equiv f(\widehat{x}) \tag{23}$$

The fluctuationlessness approximation of $\widehat{f}$ can be given as

$$\widehat{f} \approx f\left(\widehat{x}_{res}\right) \equiv f\left(P^{(n)}\widehat{x}P^{(n)}\right). \tag{24}$$

Previously we mentioned that the matrix representation of $\widehat{x}_{res}$ operator in a space spanned by the basis functions $u_1(x), \ldots, u_n(x)$ was $\mathbf{X}^{(n)}$. In that case we can say that the matrix representation of $\widehat{x}_{res}^m$ operator is $\mathbf{X}^{(n)^m}$ in this space, where $m = 0, 1, 2, \ldots$, which can be proven by induction [20]. Therefore if we represent the matrix representation of $\widehat{f}$ operator as $\mathbf{F}^{(n)}$ in this space, then the fluctuationlessness approximation for $\mathbf{F}^{(n)}$ can be written as,

$$\mathbf{F}^{(n)} \approx f\left(\mathbf{X}^{(n)}\right). \tag{25}$$

This approximation is called as Fluctuationlessness Theorem and can be stated as follows: [20]

Theorem 1 *The matrix representation of an algebraic function type operator is equal to the image of the matrix representation of the independent variable over an n dimensional space $\mathcal{H}_n$ under the function of the algebraic operator, which is analytic on $[a, b]$ at the fluctuationless limit.*

Therefore we can construct the matrix representation of a univariate function's operator via $\mathbf{X}^{(n)}$ [20–23]. In order to accomplish this work, we take the following equations:

$$\mathbf{X}^{(n)}\mathbf{x}_j = \xi_j \mathbf{x}_j, \qquad 1 \le j \le n. \tag{26}$$

Here $\mathbf{x}_j$'s are eigenvectors with norms 1, and ξ_j's are the corresponding eigenvalues. In the case when $\mathbf{X}^{(n)}$ is symmetric, eigenvectors form an orthonormal basis set in an n-dimensional space $\mathcal{K}_n$, which is constructed by cartesian vectors. We define a matrix $\mathbf{Q}$ which takes each eigenvector as columns:

$$\mathbf{Q} = [\mathbf{x}_1 \ldots \mathbf{x}_n] \tag{27}$$

The transpose of this matrix is equal to its inverse. The multiplication of $\mathbf{X}^{(n)}$ by $\mathbf{Q}$ from right hand side results in a multiplication of the matrices $\mathbf{Q}$ and a matrix whose diagonal elements are the eigenvalues ξ_i, $i = 1, \ldots, n$. We can obtain the spectral representation of the matrix $\mathbf{X}^{(n)}$ from this relation as follows:

$$\begin{aligned}
\mathbf{X}^{(n)} &= \mathbf{Q}\left[\,\xi_1\mathbf{x}_1 \;\ldots\; \xi_n\mathbf{x}_n\,\right]\mathbf{Q}^T \\
\mathbf{X}^{(n)} &= \sum_{j=1}^{n} \xi_j \mathbf{x}_j \mathbf{x}_j^T.
\end{aligned} \tag{28}$$

If we take the square of both sides in this equation, we will have the following equation for $\mathbf{X}^{(n)^2}$:

$$\begin{aligned}\mathbf{X}^{(n)^2} &= \left(\sum_{j=1}^{n} \xi_j \mathbf{x}_j \mathbf{x}_j^T\right)^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_j \xi_k \mathbf{x}_j \mathbf{x}_j^T \mathbf{x}_k \mathbf{x}_k^T \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_j \xi_k \delta_{jk} \mathbf{x}_j \mathbf{x}_k^T = \sum_{j=1}^{n} \xi_j^2 \mathbf{x}_j \mathbf{x}_j^T \end{aligned} \tag{29}$$

When we generalize this situation to any order k we can write

$$\mathbf{X}^{(n)^k} = \sum_{j=1}^{n} \xi_j^k \mathbf{x}_j \mathbf{x}_j^T, \qquad k \geq 1, \tag{30}$$

which can easily be proven by induction. In this equation the term $\mathbf{x}_j \mathbf{x}_j^T$ is an outer product and it has a projection property which projects from $\mathcal{K}_n$ to the space spanned by $\mathbf{x}_j$. These one dimensional spaces spanned by $\mathbf{x}_j$ $(j = 1, 2, \ldots, n)$ are orthogonal to each other and their union gives $\mathcal{K}_n$. Therefore the summation of n outer product of each projection matrix is a unit matrix of $\mathcal{K}_n$; that is

$$\mathbf{I} = \sum_{j=1}^{n} \mathbf{x}_j \mathbf{x}_j^T. \tag{31}$$

In this case we can rewrite (30) as

$$\mathbf{X}^{(n)^k} = \sum_{j=1}^{n} \xi_j^k \mathbf{x}_j \mathbf{x}_j^T, \qquad k \geq 0. \tag{32}$$

This means

$$f\left(\mathbf{X}^{(n)}\right) = \sum_{j=1}^{n} f\left(\xi_j\right) \mathbf{x}_j \mathbf{x}_j^T, \qquad k \geq 0. \tag{33}$$

Here $f\left(\mathbf{X}^{(n)}\right)$ is the image of the independent variable's matrix representation. By Fluctuationlessness Theorem this term is equal to the matrix representation of $\widehat{f}$ operator, which is defined on the n-dimensional Hilbert Space $\mathcal{H}_n$ of analytic and square integrable functions. If we represent $\mathbf{M}^n\left(\widehat{f}\right)$ as the matrix representation of $\widehat{f}$ operator, then the following equation holds.

$$\mathbf{M}^n\left(\widehat{f}\right) = f\left(\mathbf{X}^{(n)}\right) \tag{34}$$

3 Numerical solution of nth order initial value problems by Fluctuationlessness theorem in a closed interval [a,b]

We consider the following nth order Ordinary Differential Equation with initial conditions in a closed interval.

$$
\begin{aligned}
& y^{(n)}(t)+b_1(t)\,y^{(n-1)}(t)+\cdots+b_{n-1}(t)\,y'(t)+b_n(t)\,y(t)=r(t)\,,\\
& y(a)=y_0,\quad y'(a)=y_1,\ldots y^{(n-1)}(a)=y_{n-1},
\end{aligned}
\tag{35}
$$

where $t\in[a,b]$. We study on a subspace $\mathcal{H}_m$ of Hilbert Space $\mathcal{H}$, where $m>n$. By taking a new independent variable

$$
x\equiv(t-a)/(b-a)\,,
\tag{36}
$$

the domain is transformed to $0\le x\le 1$, and by using z as the new dependent variable (35) becomes

$$
\begin{aligned}
& z^{(n)}(x)+a_1(x)\,z^{(n-1)}(x)+\cdots+a_{n-1}(x)\,z'(x)+a_n(x)\,z(x)=q(x)\,,\\
& z(0)=z_0,\quad z'(0)=z_1,\ldots z^{(n-1)}(0)=z_{n-1},
\end{aligned}
\tag{37}
$$

where $x\in[0,1]$. The relationships between a_k and b_k in addition of the functions $r(t)$ and $q(x)$ can be stated as follows:

$$
a_k(x)=(b-a)^k\,b_k([b-a]\,x+a)\,,\quad 1\le k\le n
\tag{38}
$$

$$
q(x)=(b-a)^n\,r([b-a]\,x+a)
\tag{39}
$$

The initial conditions in (35) can be rewritten in terms of the new variables as

$$
z_k=\frac{(b-a)^k}{k!}y_k,\quad 1\le k\le n-1.
\tag{40}
$$

We take the space of the square integrable functions as the Hilbert Space $\mathcal{H}$, and define the inner product of two functions f, g in this space as

$$
(f,g)\equiv\int_0^1 f(x)\,g(x)\,w(x)\,dx,
\tag{41}
$$

where $w(x)$ is a weight function and normalized as follows:

$$
\int_0^1 w(x)\,dx=1
\tag{42}
$$

In this space we define an orthonormal set of functions as

$$\mathcal{U} = \{u_i(x)\}_{i=1}^{\infty}. \tag{43}$$

The functions in this set are constructed via Gram-Schmidt orthogonalization process. The first element in (43) has to be

$$u_1(x) = 1. \tag{44}$$

Instead of using the independent variable x, we will use the algebraic multiplication operator $\widehat{x}$ which multiplies its operand by the independent variable x, throughout the work. The spectrum of this operator is the closed interval $[0, 1]$. Therefore it has a continuous spectrum and there is no multiplicity in any eigenvalues. For the functions $a_k(x)$, $z(x)$, $z^{(i)}(x)$ and $q(x)$ we will respectively use $\widehat{a}_k$, $\widehat{z}$, $\widehat{z}^{(i)}$ and $\widehat{q}$ which multiply their operands by $a_k(x)$, $z(x)$, $z^{(i)}(x)$ and $q(x)$, respectively. Their spectrums are the continuous intervals of $\left[a_k(x)_{\min}, a_k(x)_{\max}\right]$, $\left[z(x)_{\min}, z(x)_{\max}\right]$, $\left[z^{(i)}(x)_{\min}, z^{(i)}(x)_{\max}\right]$ and $\left[q(x)_{\min}, q(x)_{\max}\right]$, respectively. They may have multiple eigenvalues depending on the structure of the functions. The operator $\widehat{z}^{(i)}$ is used for the i-th derivative of $\widehat{z}$ which multiplies its operand by $z^{(i)}(x)$. The ordinary differential equation in (37) can be expressed in terms of the images of u_1 under all these operators mentioned above as follows:

$$\left(\widehat{z}^{(n)} + \widehat{a}_1\widehat{z}^{(n-1)} + \cdots + \widehat{a}_{n-1}\widehat{z}' + \widehat{a}_n\widehat{z}\right) u_1(x) = \widehat{q}u_1 \tag{45}$$

We can express the equation (45) in corresponding cartesian space by changing each operator by its matrix representation and the function $u_1(x)$ by the unit cartesian vector $\mathbf{e}_1$ whose only nonzero element is located at the first position and has the value of 1. In other words, let $\mathbf{M}(\widehat{g})$ denote the matrix representation of the operator $\widehat{g}$, then we can write the following equation:

$$\left[\mathbf{M}\left(\widehat{z}^{(n)}\right) + \mathbf{M}(\widehat{a}_1)\,\mathbf{M}\left(\widehat{z}^{(n-1)}\right) + \cdots + \mathbf{M}(\widehat{a}_n)\,\mathbf{M}(\widehat{z})\right]\mathbf{e}_1 = \mathbf{M}(\widehat{q})\,\mathbf{e}_1, \tag{46}$$

Here the vectors and the matrices are of infinite dimension. In order to work on finite dimensional entities we will use an m-dimensional subspace $\mathcal{H}_m$ of $\mathcal{H}$ spanned by the functions $u_1(x), u_2(x), \ldots, u_m(x)$ instead of the space $\mathcal{H}$. Therefore we need to reduce the dimension by using $m \times m$ left uppermost part of the related matrices. Hence, we can rewrite (46) as

$$\begin{bmatrix} \mathbf{M}^{(m)}\left(\widehat{z}^{(n)}\right) + \mathbf{M}^{(m)}(\widehat{a}_1)\,\mathbf{M}^{(m)}\left(\widehat{z}^{(n-1)}\right) \\ + \cdots + \mathbf{M}^{(m)}(\widehat{a}_n)\,\mathbf{M}^{(m)}(\widehat{z}) \end{bmatrix}\mathbf{e}_1^{(m)} = \mathbf{M}^{(m)}(\widehat{q})\,\mathbf{e}_1^{(m)}, \tag{47}$$

If we denote the matrix representation of the operator $\widehat{x}$ on the subspace $\mathcal{H}_m$ as $\mathbf{X}^{(m)}$, then we can write the following approximated expressions by using the Fluctuationlessness Theorem.

$$\mathbf{M}^{(m)}\left(\widehat{z}^{(i)}\right) \approx z^{(i)}\left(\mathbf{X}^{(m)}\right)$$
$$\mathbf{M}^{(m)}\left(\widehat{a}_k\right) \approx a_k\left(\mathbf{X}^{(m)}\right) \tag{48}$$
$$\mathbf{M}^{(m)}\left(\widehat{q}\right) \approx q\left(\mathbf{X}^{(m)}\right)$$

The matrix $\mathbf{X}^{(m)}$ is symmetric and its spectral representation can be written as follows:

$$\mathbf{X}^{(m)} = \sum_{i=1}^{m} \xi_i \mathbf{x}_i \mathbf{x}_i^T \tag{49}$$

Here $\mathbf{x}_i$ is the eigenvector with a unit norm of the ith eigenvalue ξ_i. Substituting (47) in (48) we obtain the following result.

$$\sum_{i=1}^{m} \left[\begin{array}{c} z^{(n)}\left(\xi_i\right) + a_1\left(\xi_i\right) z^{(n-1)}\left(\xi_i\right) \\ + \cdots + a_n\left(\xi_i\right) z\left(\xi_i\right) - q\left(\xi_i\right) \end{array} \right] \left(\mathbf{x}_i^T \mathbf{e}_1^{(m)}\right) \mathbf{x}_i = 0 \tag{50}$$

Since the eigenvectors are linearly independent the coefficients of $\mathbf{x}_i$ should vanish. So we can write the following equations:

$$z^{(n)}\left(\xi_i\right) + a_1\left(\xi_i\right) z^{(n-1)}\left(\xi_i\right) + \cdots + a_n\left(\xi_i\right) z\left(\xi_i\right) - q\left(\xi_i\right) = 0, \quad 1 \le i \le m. \tag{51}$$

We propose the following structure for the approximated solution of the differential equation:

$$f\left(x\right) = z_0 + z_1 x + \cdots + z_{n-1} x^{n-1} + \sum_{k=1}^{m-n+1} f_k x^{k+n-1} \tag{52}$$

To find the unknown constants f_i, $(1 \le i \le m-n+1)$ in (52) we construct a set of $[m-n+1]$-dimensional vectors and matrices as follows:

$$\mathbf{K}^{(i,j)} = \begin{cases} a_n\left(\xi_{n-1+i}\right) \xi_{n-1+i}, & i = j \\ 0, & i \ne j \end{cases},$$

$$\mathbf{K}_{\mathbf{r}}^{(i,j)} = \begin{cases} a_{n-r}\left(\xi_{n-1+i}\right), & i = j \\ 0, & i \ne j \end{cases},$$

$$\mathbf{S}_{\mathbf{r}}^{(i,j)} = \begin{cases} \prod_{k=1}^{r} \left(n+i-k\right), & i = j \\ 0, & i \ne j \end{cases},$$

$$
\begin{aligned}
\mathbf{V}_r &= \begin{bmatrix} \xi_n^{n-r} & \xi_n^{n-r+1} & \dots & \xi_n^{m-r} \\ \xi_{n+1}^{n-r} & \xi_{n+1}^{n-r+1} & \dots & \xi_{n+1}^{m-r} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_m & \xi_m^{n-r+1} & \dots & \xi_m^{m-r} \end{bmatrix}, \\
\mathbf{a_n}^T &= \left[a_n\left(\xi_n\right) \dots a_n\left(\xi_m\right) \right], \\
\mathbf{u}^T &= \left[1\ 0 \dots 0 \right], \\
\mathbf{q}^T &= \left[q\left(\xi_n\right) \dots q\left(\xi_m\right) \right], \\
\mathbf{a} &= z_0\mathbf{a_n} + \sum_{i=1}^{n-1} z_i \begin{bmatrix} \mathbf{K}.\mathbf{V}_{n-i+1} + i\mathbf{K}_1\mathbf{V}_{n-i+1} + i\left(i-1\right)\mathbf{K}_2\mathbf{V}_{n-i+2} \\ +i\left(i-1\right)\left(i-2\right)\mathbf{K}_3\mathbf{V}_{n-i+3} + \dots + i!\mathbf{K}_i\mathbf{V}_n \end{bmatrix} \cdot \mathbf{u}
\end{aligned} \tag{53}
$$

By using these matrices and vectors the unknown coefficients f_i, $(1 \le i \le m-n+1)$ can be found by

$$
\mathbf{f} = \left(\mathbf{K}\mathbf{V}_1 + \sum_{r=1}^{n-1} \mathbf{K}_r\mathbf{V}_r\mathbf{S}_r + \mathbf{V}_n\mathbf{S}_n \right)^{-1} \cdot (\mathbf{q} - \mathbf{a}). \tag{54}
$$

[24]. Hence we obtain the numerical solution of (37). By making the transformation of the independent variable x by t defined in (36), we obtain the numerical solution of the initial value problem (35) in $[a, b]$.

4 Fluctuationlessness theorem in boundary value problems

We consider the following Boundary Value Problem (BVP) and its boundary conditions:

$$
y'' + p\left(x\right)y' + q\left(x\right)y = r\left(x\right), \tag{55}
$$

$$
a_1 y\left(0\right) + a_2 y'\left(0\right) = c, \tag{56}
$$

$$
b_1 y\left(1\right) + b_2 y'\left(1\right) = d, \tag{57}
$$

where $0 < x < 1$. In the case when the values for $r\left(x\right)$, c and d are all 0, the problem is a linear homogenous BVP. If we assume $y = \phi\left(x\right)$ is a nontrivial solution, then for any constant k, the function $y = k\phi\left(x\right)$ is also a solution because of the linear and homogenous structure of the problem. Therefore there are infinitely many solutions related to ϕ. It is also possible to show that the equations (55), (56) and (57) have at most one nontrivial solution. But a more general second order BVP may have a solution set related to linearly independent solutions, say ϕ_1 and ϕ_2 [2].

In this paper we will concentrate on the BVPs that is presented in the equations (55), (56) and (57). This problem have a unique solution. To find this solution numerically, we propose the following structure:

$$f\left(x\right)=\sum_{k=0}^{n}f_{k}x^{k} \tag{58}$$

If we substitute this solution in the equations (55), (56) and (57), we obtain the following equations:

$$\begin{aligned}r\left(x\right)=&\,q\left(x\right)f_{0}+\left(q\left(x\right)x+p\left(x\right)\right)f_{1}+\left(q\left(x\right)x^{2}+2p\left(x\right)x+2\right)f_{2}\\&+\left(q\left(x\right)x^{3}+3p\left(x\right)x^{2}+6x\right)f_{3}\\&+\cdots+\left(q\left(x\right)x^{n}+np\left(x\right)x^{n-1}+n\left(n-1\right)x^{n-2}\right)f_{n}\end{aligned} \tag{59}$$

$$c=a_{1}f_{0}+a_{2}f_{1} \tag{60}$$

$$d=b_{1}f_{0}+\left(b_{1}+b_{2}\right)f_{1}+\sum_{k=2}^{n}\left(b_{1}+kb_{2}\right)f_{k} \tag{61}$$

For the numerical solution we will apply Fluctuationlessness Theorem in the interval [0, 1]. We write the following equation for (55) in terms of operators as follows:

$$\mathbf{M}^{(n)}\left(\widehat{r}\right)\mathbf{e}_{1}^{(n)}=\left[\mathbf{M}^{(n)}\left(\widehat{f}''\right)+\mathbf{M}^{(n)}\left(\widehat{p}\right)\mathbf{M}^{(n)}\left(\widehat{f}'\right)+\mathbf{M}^{(n)}\left(\widehat{q}\right)\mathbf{M}^{(n)}\left(\widehat{f}\right)\right]\mathbf{e}_{1}^{(n)}, \tag{62}$$

where $\mathbf{M}^{(n)}\left(\widehat{f}\right)$ stands for the matrix representation of the operator $\widehat{f}$ in the Hilbert Space $\mathcal{H}_{n}$. By Fluctuationlessness Theorem we know that

$$\mathbf{M}^{(n)}\left(\widehat{f}\right)\approx f\left(\mathbf{X}^{(n)}\right). \tag{63}$$

The matrix $\mathbf{X}^{(n)}$ is symmetric and its spectral representation can be written as follows:

$$\mathbf{X}^{(n)}=\sum_{i=1}^{n}\xi_{i}\mathbf{x}_{i}\mathbf{x}_{i}^{T} \tag{64}$$

Here ξ_i is the ith eigenvalue and $\mathbf{x}_i$ is its eigenvector with a unit norm. Substituting (63) and (64) in (62) we obtain the following result:

$$\sum_{i=1}^{n}\left[f''\left(\xi_{i}\right)+p\left(\xi_{i}\right)f'\left(\xi_{i}\right)+q\left(\xi_{i}\right)f\left(\xi_{i}\right)-r\left(\xi_{i}\right)\right]\left(\mathbf{x}_{i}^{T}\mathbf{e}_{1}^{(n)}\right)\mathbf{x}_{i}=0 \tag{65}$$

Since the eigenvectors are linearly independent, then the coefficients of $\mathbf{x}_i$ equal zero. So we can write the following equations:

$$f''\left(\xi_{i}\right)+p\left(\xi_{i}\right)f'\left(\xi_{i}\right)+q\left(\xi_{i}\right)f\left(\xi_{i}\right)-r\left(\xi_{i}\right)=0,\quad i=1,\ldots,n \tag{66}$$

To find the unknown constants f_i, $(2 \le i \le n)$ in (59) we construct a set of vectors and matrices as follows:

$$\begin{aligned}
\mathbf{K}_{1(i,j)} &= \begin{cases} q(\xi_{i+1})\,\xi_{i+1}, & i = j \\ 0, & i \neq j \end{cases},\\
\mathbf{K}_{2(i,j)} &= \begin{cases} p(\xi_{i+1}), & i = j \\ 0, & i \neq j \end{cases},\\
\mathbf{K}_{3(i,j)} &= \begin{cases} i+1, & i = j \\ 0, & i \neq j \end{cases},\\
\mathbf{K}_{4(i,j)} &= \begin{cases} i(i+1), & i = j \\ 0, & i \neq j \end{cases},\\
\mathbf{V}_1 &= \begin{bmatrix} \xi_2 & \xi_2^2 & \dots & \xi_2^{n-1} \\ \xi_3 & \xi_3^2 & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_n & \xi_n^2 & \dots & \xi_n^{n-1} \end{bmatrix},\\
\mathbf{V}_2 &= \begin{bmatrix} 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^{n-2} \\ 1 & \xi_3 & \xi_3^2 & \dots & \xi_3^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_n & \xi_n^2 & \dots & \xi_n^{n-2} \end{bmatrix},\\
\mathbf{a}_{(i,1)} &= \left[q(\xi_{i+1})\,f_0 + \left(p(\xi_{i+1}) + q(\xi_{i+1})\,\xi_{i+1}\right) f_1\right],\\
\mathbf{r}^T &= \left[\, r(\xi_2) \dots r(\xi_n) \,\right],\\
\mathbf{f}^T &= \left[\, f_2 \dots f_n \,\right].
\end{aligned} \tag{67}$$

We can write the equation (59) in terms of these matrices and vectors as follows:

$$\mathbf{r} = (\mathbf{K}_1\mathbf{V}_1 + \mathbf{K}_2\mathbf{V}_1\mathbf{K}_3 + \mathbf{V}_2\mathbf{K}_4)\,\mathbf{f} + \mathbf{a} \tag{68}$$

Since the eigenvalues are distinct, the matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ are invertible. Therefore we can find $\mathbf{f}$ by the following formula:

$$\mathbf{f} = (\mathbf{K}_1\mathbf{V}_1 + \mathbf{K}_2\mathbf{V}_1\mathbf{K}_3 + \mathbf{V}_2\mathbf{K}_4)^{-1}\,(\mathbf{r} - \mathbf{a}) \tag{69}$$

When we substitute the values of $\mathbf{f}$ in the equation (58) we obtain the following equation.

$$f(x) = f_0 + f_1 x + \sum_{k=2}^{n} f_k x^k \tag{70}$$

Here all the values except f_0 and f_1 are known. To find f_0 and f_1, the equations (60), (61) and (70) can be solved together. Hence we obtain the numerical solution [25].

5 Numerical solution of ordinary differential equations with eigenvalue problems

We consider the problem consisting of the differential equation,

$$y'' + \lambda y = 0, \tag{71}$$

together with the boundary conditions

$$y(0) = 0, \quad y(1) = 0. \tag{72}$$

The analytical solution of the problems (71) and (72) depends on the values of the unknown λ. If $\lambda = 0$, then there is only trivial solution $\phi(x) = 0$. In the case of $\lambda < 0$, the solution is again trivial, i.e: $\phi(x) = 0$. The last case occurs when $\lambda > 0$. The solutions exists when

$$\lambda = (k\pi)^2, \quad k = 1, \ldots, n, \ldots \tag{73}$$

The numbers λ are called eigenvalues. The eigenfunctions associated with these eigenvalues are called eigenfunctions and these functions are $\phi_n(x) = c_n \sin n\pi x$.

We will obtain the numerical solution to the problems (71) and (72) by Fluctuationlessness Theorem. We again propose the same structure for the numerical solution:

$$f(x) = \sum_{k=0}^{n} f_k x^k \tag{74}$$

When we substitute this solution to the equations (71) and (72), we will have the following equations:

$$\lambda f_0 + (\lambda x) f_1 + \left(2 + \lambda x^2\right) f_2 + \cdots + \left(n(n-1)x^{n-2} + \lambda x^n\right) f_n = 0 \tag{75}$$

$$f(0) = f_0 = 0 \tag{76}$$

$$f(1) = \sum_{k=1}^{n} f_k = 0 \tag{77}$$

We obtain the following equation for f_1 by (77):

$$f_1 = -\sum_{k=2}^{n} f_k \tag{78}$$

The equations (78) and (75) form the following equation finally:

$$\begin{aligned} 0 = \; & \left(2 + \lambda x^2 - \lambda x\right) f_2 + \left((3)(2) + \lambda x^3 - \lambda x\right) f_3 \\ & + \cdots + \left(n(n-1)x^{n-2} + \lambda x^n - \lambda x\right) f_n \end{aligned} \tag{79}$$

By Fluctuationlessness Theorem we obtain the following equations:

$$\begin{aligned}
\left(2+\lambda\xi_1^2-\lambda\xi_1\right) f_2+\cdots+\left(n\,(n-1)\,\xi_1^{n-2}+\lambda\xi_1^n-\lambda\xi_1\right) f_n &= 0\\
\left(2+\lambda\xi_2^2-\lambda\xi_2\right) f_2+\cdots+\left(n\,(n-1)\,\xi_2^{n-2}+\lambda\xi_2^n-\lambda\xi_2\right) f_n &= 0\\
&\vdots\\
\left(2+\lambda\xi_n^2-\lambda\xi_n\right) f_2+\cdots+\left(n\,(n-1)\,\xi_n^{n-2}+\lambda\xi_n^n-\lambda\xi_n\right) f_n &= 0
\end{aligned}$$

We will use the following matrices and vectors to find the unknown values f_k, $k = 2, \ldots, n$:

$$\begin{aligned}
\mathbf{A}_0 &= \begin{bmatrix} 2 & 6\xi_1 & 12\xi_1^2 & \ldots & n\,(n-1)\,\xi_1^{n-2}\\ 2 & 6\xi_2 & 12\xi_2^2 & \ldots & n\,(n-1)\,\xi_2^{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 2 & 6\xi_n & 12\xi_n^2 & \ldots & n\,(n-1)\,\xi_n^{n-2}\end{bmatrix}_{(n\times n-1)},\\
\mathbf{A}_1 &= \begin{bmatrix} -\xi_1+\xi_1^2 & -\xi_1+\xi_1^3 & \ldots & -\xi_1+\xi_1^n\\ -\xi_2+\xi_2^2 & -\xi_2+\xi_2^3 & \ldots & -\xi_2+\xi_2^n\\ \vdots & \vdots & \vdots & \vdots\\ -\xi_n+\xi_n^2 & -\xi_n+\xi_n^3 & \ldots & -\xi_n+\xi_n^n\end{bmatrix}_{(n\times n-1)},\\
\mathbf{f}^T &= \left[\, f_2 \ \ldots \ f_n \,\right]_{(n-1\times 1)}
\end{aligned} \tag{80}$$

Therefore (75) can be expressed by these matrices and vectors as follows:

$$(\mathbf{A}_0+\lambda\mathbf{A}_1)\,\mathbf{f}=\mathbf{0} \tag{81}$$

In this system of equations, there are n equations and $n-1$ unknown f_i values. This situation occurs since f_1 is written in terms of unknown f_i values, where $i = 2, \ldots, n$. Also the matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ are rectangular matrices. To make this system balanced, we multiply the equation (81) by $\mathbf{A}_1^T$ from left. The new equation becomes

$$\left(\mathbf{A}_1^T\mathbf{A}_0+\lambda\mathbf{A}_1^T\mathbf{A}_1\right)\mathbf{f}=\mathbf{0}. \tag{82}$$

We rename the above matrices as $\mathbf{B}_0 = \mathbf{A}_1^T\mathbf{A}_0$ and $\mathbf{B}_1 = \mathbf{A}_1^T\mathbf{A}_1$, so the equation (82) becomes

$$(\mathbf{B}_0+\lambda\mathbf{B}_1)\,\mathbf{f}=\mathbf{0}. \tag{83}$$

We define a new vector $\mathbf{g}$ as follows:

$$\mathbf{g}=\mathbf{B}_1\mathbf{f} \tag{84}$$

Since the eigenvalues of $\mathbf{X}^{(n)}$ in the space of $\mathcal{H}_n$ are distinct, then $\mathbf{B}_1$ is invertible. Therefore $\mathbf{f}$ can be written as

$$\mathbf{f} = \mathbf{B}_1^{-1}\mathbf{g}. \tag{85}$$

If we substitute (85) in (82) we obtain the following equation:

$$\left(-\mathbf{B}_0\mathbf{B}_1^{-1} - \lambda\mathbf{I}\right)\mathbf{g} = \mathbf{0} \tag{86}$$

Here λ values are the eigenvalues of the matrix $-\mathbf{B}_0\mathbf{B}_1^{-1}$and $\mathbf{g}$ vector is the eigenvectors. Once we obtain $\mathbf{g}$ vector, we can calculate $\mathbf{f}$ vector by (85).

6 Implementation

In this section we give some numerical examples related to initial value problems with different orders starting the simplest case.

Example 1 The first problem is

$$y'(x) - y(x) = 0, \quad y(0) = 1, \tag{87}$$

over the interval [0, 1]. The analytical solution of this problem is $y(x) = e^x$. For the numerical solution we propose the following structure:

$$f(x) = f_0 + \sum_{k=1}^{n} f_k x^k \tag{88}$$

Table 1 shows the numerical results of the absolute error of the solutions obtained by Fluctuationlessness Theorem and MacLaurin series of the analytical solution on varying number of nodes. The error is calculated by the following equation.

$$e = \left(\sum_{k=1}^{n} |f(x_k) - y(x_k)|^2\right)^{1/2}, \tag{89}$$

where x_k is the kth node of the given interval divided equally to n nodes.

We observe from this table, the numerical solution by using Fluctuationlessness Theorem gives better results than MacLaurin Series even by using less nodes. The series method showed approximately same order of error after using 20 nodes.

Example 2 Table 2 shows the results for some differential equations of the form

$$y'(x) + a(x)y(x) = 0, \qquad y(0) = 1, \tag{90}$$

where $a(x)$ is taken as 5, −5, 10, −10, 20, −20 in each of the problem.

Table 1 Comparison of the analytical solution of 1-st example with numerical solutions on varying nodes

n	Abs.err. with fluctuationlessness Thm.	Abs err. with MacLaurin series
5	2.3362×10^{-6}	0.00041637
7	3.485×10^{-9}	8.18972×10^{-6}
10	5.34472×10^{-14}	8.81818×10^{-9}
20	9.15513×10^{-16}	9.15513×10^{-16}
50	1.4895×10^{-15}	1.4895×10^{-15}

Table 2 Comparison of the analytical solutions of ODEs with numerical solution when $n = 20$

$a\,(x)$	Analytic soln.	Abs. err. with fluc. Thm.	Abs. err. with MacLaurin series
-5	$y = e^{5x}$	4.83756×10^{-14}	4.2574×10^{-6}
5	$y = e^{-5x}$	3.3950×10^{-15}	2.75737×10^{-6}
-10	$y = e^{10x}$	6.7711×10^{-9}	12.0693
10	$y = e^{-10x}$	4.26348×10^{-13}	4.9011
-20	$y = e^{20x}$	97.80376	6.5587×10^{7}
20	$y = e^{-20x}$	1.0312×10^{-8}	7.8596×10^{6}

We can clearly see from this table, numerical solution obtained by Fluctuationlessness Theorem gives better results than the series solution. In Table 3 the errors are showed by using 50 nodes.

We observe that as the power of exponential functions increases in absolute value, the smoothness of the curve decreases. Therefore the numerical solution moves away from the exact solution depending on the power of $\exp(x)$. And we obtain better results in larger number of nodes.

The next example is for the higher order ODE.

Example 3 We consider the following third order ODE with given initial values:

$$\begin{aligned} &y''' + 4y' = x \\ &y\,(0) = 0, \quad y'\,(0) = 0, \quad y''\,(0) = 1 \end{aligned} \tag{91}$$

The analytical solution of this problem is

$$y(x) = \frac{3}{16}\,(1 - \cos 2x) + \frac{1}{8}x^2. \tag{92}$$

When we solve this problem in $\mathcal{H}_{10}$ by Fluctuationlessness Theorem, we obtain the error as 1.8971×10^{-9}. The error of the analytical solution with its series expansion is calculated as 1.6167×10^{-6}. Therefore the numerical solution by Fluctuationlessness Theorem achieved a good approximation in Example 3.

Table 3 Comparison of the analytical solutions of ODEs with numerical solution when $n = 50$

$a\,(x)$	Analytic soln.	Abs. err. with fluc. Thm.	Abs. err. with MacLaurin series
-5	$y = e^{5x}$	1.4352×10^{-13}	1.4352×10^{-13}
5	$y = e^{-5x}$	1.1618×10^{-14}	1.1618×10^{-14}
-10	$y = e^{10x}$	2.2392×10^{-11}	2.2392×10^{-11}
10	$y = e^{-10x}$	1.1846×10^{-12}	1.1846×10^{-12}
-20	$y = e^{20x}$	9.9316×10^{-5}	0.87991
20	$y = e^{-20x}$	5.1667×10^{-9}	0.40115

Table 4 Comparison of the analytical solution of 4th example with numerical solutions on varying nodes

n	Abs.err. with F.T.	Abs err. with MacLaurin series
10	1.19907×10^{-9}	9.62475×10^{-7}
20	2.1524×10^{-25}	2.46558×10^{-17}
50	3.04210×10^{-84}	2.699×10^{-59}

Example 4 This example is a Boundary Value Problem given by the following:

$$\begin{aligned} &y'' + 2y = -x \\ &y(0) = 0, \quad y(1) + y'(1) = 0 \end{aligned} \tag{93}$$

The analytical solution of this problem is calculated as

$$y(x) = \frac{-\sqrt{2}x\cos\sqrt{2} - x\sin\sqrt{2} + 2\sin\sqrt{2}x}{2\left(\sqrt{2}\cos\sqrt{2} + \sin\sqrt{2}\right)}. \tag{94}$$

In Table 4, the errors obtained by the numerical solution with Fluctuationlessness Theorem and the series expansion of the analytical solution is presented:

We can observe from this table that the numerical solution by the developed method has a better approximation even by using not too many mesh points.

Example 5 The last example we will study is a Boundary Value Problem containing an unknown parameter, which is an eigenvalue problem given by

$$y'' + \lambda y = 0, \tag{95}$$

together with the boundary conditions

$$y\,(0) = 0, \quad y\,(1) = 0. \tag{96}$$

We will study in $\mathcal{H}_{10}$. When we apply the Fluctuationlessness Theorem we obtain the eigenvalues as

$$\lambda_1 = 9.87, \quad \lambda_2 = 39.48, \quad \lambda_3 = 88.83, \quad \lambda_4 = 158.12, \quad \lambda_5 = 248.45,$$
$$\lambda_6 = 398.22, \quad \lambda_7 = 618.841, \quad \lambda_8 = 2302.17, \quad \lambda_9 = 4974.02. \tag{97}$$

The eigenvalues obtained by analytically are calculated as

$$\lambda_1 = 9.8696, \quad \lambda_2 = 39.4784, \quad \lambda_3 = 88.8264, \quad \lambda_4 = 157.914, \quad \lambda_5 = 246.74,$$
$$\lambda_6 = 355.306, \quad \lambda_7 = 483.611, \quad \lambda_8 = 631.655, \quad \lambda_9 = 799.438. \tag{98}$$

The numerical solution is found by the eigenvectors of $-\mathbf{B}_0\mathbf{B}_1^{-1}$. We represent the eigenvectors associated with these eigenvalues as $\mathbf{u}_k$, $(k = 1, \ldots, 9)$. If we write the coefficients of the eigenfunctions as a vector $\mathbf{v}_k$, this vector is calculated by as

$$\mathbf{v}_k = \mathbf{B}_1^{-1}\mathbf{u}_k, \qquad k = 1, \ldots, 9. \tag{99}$$

We denote the entries of the vectors $\mathbf{v}_k$ as $h_j^{(k)}$ and the first coefficient h_1 is found by

$$h_1^{(k)} = -\sum_{j=1}^{n-1} \mathbf{v}_k^{(j,1)}. \tag{100}$$

Other coefficients are calculated as

$$h_j^{(k)} = \mathbf{v}_k^{(j,1)}. \tag{101}$$

Therefore the numerical solution can be obtained as

$$\phi_k(x) = h_1^{(k)} x + \sum_{j=1}^{n-1} h_j^{(k)} x^{j+1}. \tag{102}$$

The analytical solution of this problem can be written as follows:

$$y_n(x) = c_n \sin(n\pi x), \quad n = 1, \ldots, 9 \tag{103}$$

The coefficients c_n can be different in these functions or some of them may be equal. In order to find these coefficients we normalize each eigenfunction in the unit interval as

$$\int_0^1 (c_n \sin[n\pi x])^2 \, dx = 1. \tag{104}$$

Table 5 Comparison of the eigenfunctions of 5th example with numerical solutions

Eigenfunctions	Abs. err. by fluc. Thm.	Abs. err. with Maclaurin series
$\phi_1(x)$	1.87648×10^{-9}	0.0103135
$\phi_2(x)$	3.50988×10^{-5}	17.7818
$\phi_3(x)$	6.99729×10^{-4}	1194.96
$\phi_4(x)$	0.0416281	21135.1
$\phi_5(x)$	0.146869	183194
$\phi_6(x)$	0.817106	1.03076×10^6
$\phi_7(x)$	0.900223	4.35423×10^6
$\phi_8(x)$	3.15304	1.50009×10^7
$\phi_9(x)$	4.56265	4.43658×10^7

We obtain the coefficients c_n as $-\sqrt{2}$ and $\sqrt{2}$. We apply the same procedure for the numerical solutions:

$$\int_0^1 (s_k \phi_k(x))^2 \, dx = 1, \quad k = 1, \ldots, 9 \tag{105}$$

The coefficients are different in each eigenfunction. For example for the first eigenfunction $\phi_1(x)$, s_1 is calculated as -7.9915.

The resulting errors obtained by Fluctuationlessness Theorem and the MacLaurin series expansion of each eigenfunction are presented in Table 5.

7 Conclusion

In this work we developed a new method for the numerical solution of Initial Value Problems of any order and Boundary Value Problems of Linear Ordinary Differential Equations by using Fluctuationlessness Theorem. We are interested on the problems which have analytic solutions at the given initial and boundary conditions to make a comparison of the numerical method we develop. However it must be noted that the singular solution $y = 0$ is also a solution of differential equations. Since the solutions of the problems are analytic, we can obtain their MacLaurin series expansion. The results of the numerical experiments show a remarkable convergence comparing to their series expansion at the indicated degree even we use a few number of nodes. We also observe the effect of structure of the analytic solution on the convergence.

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